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THE AXISYMMETRIC MIXED PROBLEM IN THE THEORY OF ELASTICITY FOR A HOLLOW TRUNCATED CIRCULAR CONE†

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An explicit solution is constructed for the axisymmetric problem of the stressed state of a hollow circular cone truncated by two spherical surfaces (the ends of the cone) with a normal load acting on one of the ends (the other end is unloaded) and sliding clamping or the side surfaces of the cone. A number of special cases is considered including the stressed state of a spherical cupola supported on an absolutely rigid, smooth, plane base and there can be a conical incision at the centre of the cupola. The method of solution is easily extended to the case of arbitrary axisymmetric loading of the ends and is based on the use of a new integral transformation the derivation of which is presented.

1. FORMULATION OF THE PROBLEM

An elastic (shear modulus G, Poisson's ratio μ) hollow circular cone: $a < r < b$, $\omega_0 < \theta < \omega_1$, $-\pi, < \varphi < \pi$, truncated by two spherical surfaces $r = a$ and $r = b$, is subject to the action of a normal compressive load of strength $p(\theta)$ on the surface $r = b$ while the other end of the cone ($r = a$) is assumed to be unloaded. Conditions of sliding clamping are satisfied on the side conical surfaces $\theta = \omega_0$ and $\theta = \omega_1$.

Adopting the notation

$$2Gu_r(r,\theta) = u(r,\theta),\quad 2Gu_\theta(r,\theta) = v(r,\theta) \tag{1.1}$$

$$\tau_{r\theta}(r,\theta) = \tau_{\theta r}(r,\theta) = \tau_r(\theta)$$

we write the boundary conditions

$$\sigma_r(a,\theta) = \tau_r(a,\theta) = 0,\quad \sigma_r(b,\theta) = -p(\theta),\quad \tau_r(b,\theta) = 0,\quad \omega_0 \leqslant \theta \leqslant \omega_1 \tag{1.2}$$

$$v(r,\omega_j) = 0,\quad \tau_r(r,\omega_j) = 0,\quad j = 0,1;\quad a \leqslant r \leqslant b \tag{1.3}$$

In order to solve the boundary-value problem, we shall use the solution of the Lamé equation in a form which has been proposed earlier in [1]

$$u(r,\theta) = \Phi'(r,\theta) - 2(1-\mu)r\Delta F,\quad v(r,\theta) = r^{-1}\Phi^{\cdot}(r,\theta) \tag{1.4}$$

$$\Phi(r,\theta) = rF'(r) + (3-4\mu)F,\quad \Delta\Delta F = 0$$

$$\Delta F = \frac{(r^2F')'}{r^2} - \frac{\nabla_0 F}{r^2},\quad -\nabla_m F = \frac{(\sin\theta F^{\cdot})^{\cdot}}{\sin\theta} - \frac{m^2 F}{\sin^2\theta},\quad m = 0,1,2,\ldots \tag{1.5}$$

Henceforth, a prime denotes a derivative with respect to r and a dot denotes a derivative with respect to θ. As in [2], we represent the biharmonic function F in the form

$$F(r,\theta) = r^2\Psi(r,\theta) + \Omega(r,\theta),\quad \Delta\Psi = \Delta\Omega = 0 \tag{1.6}$$

This enables us to express the strain and stress fields, according to Eqs (1.4), in terms of the two harmonic functions

$$u = -(1-2\mu)[r^3\Psi''' + 4r^2\Psi' + 2r\Psi] + 2(1-\mu)r\nabla_0\Psi + r\Omega'' + 4(1-\mu)\Omega' \tag{1.7}$$

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$$v = [r^2\Psi' + (5-4\mu)r\Psi + \Omega' + (3-4\mu)r^{-1}\Omega]^{\bullet} \tag{1.8}$$

$$\tau_r(r,\theta) = A^{\bullet}(r,\theta), \quad A(r,\theta) = \mu r^2\Psi'' + (1+2\mu)r\Psi' - (1-2\mu)\Psi + (1-\mu)\nabla_0\Psi +$$

$$+\Omega'' + (3-4\mu)(r^{-1}\Omega)' \tag{1.9}$$

$$(1-2\mu)\sigma_r(r,\theta) = -(1-2\mu)[(1-\mu)r^3\Psi''' + (7-5\mu)r^2\Psi'' + 2(5-\mu)r\Psi' +$$

$$+2(1+\mu)\Psi] + (1-\mu)r\Omega''' + (4\mu^2 - 7\mu + 5)\Omega'' + 8\mu(1-\mu)r^{-1}\Omega' +$$

$$+\nabla_0\{(2\mu^2 - 5\mu + 2)(r\Psi)' - \mu r^{-1}[\Omega' + (3-4\mu)r^{-1}\Omega]\} \tag{1.10}$$

Hence, the problem reduces to finding the harmonic functions Ψ and Ω from boundary conditions (1.2) and (1.3).

In order to construct an exact solution of this problem, it is necessary to establish a suitable integral transform and to derive an inversion formula for it. An integral transform is established below which enables us to obtain not only the solution of the problem in question but also the solutions of more-complex boundary-value problems for hollow cones, including non-axisymmetric cones.

2. DERIVATION OF THE INTEGRAL TRANSFORM

The derivation of the integral transform is based on the solution of the following Sturm–Liouville boundary-value problems

$$-\nabla_m T(\theta) - (\lambda + \tfrac{1}{4})T(\theta) = 0, \quad \omega_0 < \theta < \omega_1, \quad m = 0,1,2,\ldots$$

$$\text{a) } T(\omega_j) = 0, \quad b)\ T^{\bullet}(\omega_j) + h_j T(\omega_j) = 0, \quad c)\ T^{\bullet}(\omega_j) = 0; \quad j = 0,1 \tag{2.1}$$

It should be noted that the conical functions

$$T(\theta) = P^m_{-\frac{1}{2}+i\sqrt{\lambda}}(\cos\theta), \quad Q^m_{-\frac{1}{2}+i\sqrt{\lambda}}(\cos\theta) \tag{2.2}$$

are linearly independent solutions of the differential equation in (2.1) [2].

It is required to determine the eigenvalues λ_k (or $\nu_k = -1/2 + i\sqrt{\lambda_k}$) and the eigenfunctions of problems (2.1) and to obtain a formula for the expansion of an arbitrary function in these eigenfunctions.

In order to use the well-known apparatus in [3], we make the substitution

$$T(\theta) = (\sin\theta)^{-1/2} y(\theta) \tag{2.3}$$

in (2.1) and reduce problem (2.1) to the boundary-value problem

$$y^{\bullet\bullet}(\theta) - \{\lambda + q(\theta)\}y(\theta) = 0, \quad \omega_0 < \theta < \omega_1 \quad [q(\theta)\sin^2\theta = m^2 - \tfrac{1}{4}] \tag{2.4}$$

$$l_j y(\theta) \equiv y(\omega_j)\cos\alpha_j + y^{\bullet}(\omega_j)\sin\alpha_j = 0, \quad j = 0,1$$

which has been considered previously in [3].

Following the scheme proposed in [3] and taking account of relations (2.2) and (2.3), we start from the functions

$$\varphi_0(\theta,\lambda) = \sqrt{\sin\theta}\, P^m_\nu(\cos\theta), \quad \chi_0(\theta,\lambda) = \sqrt{\sin\theta}\, Q^m_\nu(\cos\theta) \tag{2.5}$$

$$\nu = -\tfrac{1}{2} + i\sqrt{\lambda}$$

Using formula 3.4 (25) from [4], we calculate their Wronskian

$$W(\varphi_0, \chi_0) = -\Gamma_m(\nu)$$

where

$$\Gamma_m(\nu)=\frac{2^{2m}\Gamma(1+\tfrac{1}{2}m+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{1}{2}\nu)}{\Gamma(1-\tfrac{1}{2}m+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m+\tfrac{1}{2}\nu)} \tag{2.6}$$

Next, we construct the functions

$$\begin{aligned}-\Gamma_m(\nu)\varphi(\theta,\lambda)&=\varphi_0(\theta,\lambda)l_0\chi_0-\chi_0(\theta,\lambda)l_0\varphi_0=\sqrt{\sin\theta}F^m_{\nu,0}(\theta),\\ -\Gamma_m(\nu)\chi(\theta,\lambda)&=\sqrt{\sin\theta}F^m_{\nu,1}(\theta)\end{aligned} \tag{2.7}$$

where

$$F^m_{\nu,j}(\theta)=P^m_\nu(\cos\theta)l_j\chi_0-Q^m_\nu(\cos\theta)l_j\varphi_0,\quad j=0,1 \tag{2.8}$$

and find their derivatives. The Wronskian of these functions is found to be equal to

$$W(\varphi,\chi)=W=-[l_0\varphi_0l_1\chi_0-l_1\varphi_0l_0\chi_0]/\Gamma_m(\nu) \tag{2.9}$$

Taking into account the fact that, by virtue of relation (2.3),

$$l_j\varphi_0=\sqrt{\sin\omega_j}\,l_j^*P^m_\nu,\quad l_j\chi_0=\sqrt{\sin\omega_j}\,l_j^*Q^m_\nu \tag{2.10}$$

where

$$l_j^*f(\theta)\equiv l_j^*f=\sin\alpha_jf'(\omega_j)+[\cos\alpha_j+\tfrac{1}{2}\operatorname{ctg}\omega_j\sin\alpha_j]f(\omega_j) \tag{2.11}$$

instead of (2.9), we shall have

$$\begin{aligned}\Gamma_m(\nu)W&=\sqrt{\sin\omega_0\sin\omega_1}\,\Delta^m_\nu\\ \Delta^m_\nu&=l_1^*P^m_\nu l_0^*Q^m_\nu-l_0^*P^m_\nu l_1^*Q^m_\nu\end{aligned} \tag{2.12}$$

Hence, the function $\Phi(\theta,\lambda)$, defined by formula (1.6.2) from [3], has been constructed.

The eigenvalues λ_k of problem (2.4) and the eigenvalues associated with them $\nu_k=-1/2+i\sqrt{\lambda_k}$ must be found from the equation

$$\Delta^m_{\nu_k}=0,\quad k=0,1,2,\ldots \tag{2.13}$$

The equality

$$l_0^*P^m_{\nu_k}=l_1^*P^m_{\nu_k}l_0^*Q^m_{\nu_k}[l_1^*Q^m_{\nu_k}]^{-1} \tag{2.14}$$

is a consequence of (2.13),

According to what has been previously described in [3], in order to obtain an expansion of the function $f(\theta)$ in the eigenvalues of problem (2.4), it remains to calculate the residue of the function $\Phi(\theta,f)$ when $\lambda=\lambda_k$ using the formula given in [3] and to calculate the ratio $\chi(\theta,\lambda_k)K\varphi(\theta,\lambda_k)$ Then, according to relations (2.7), (2.8) and (2.10), we obtain

$$\begin{aligned}\Gamma_m(\nu_k)\chi(\theta,\lambda_k)&=\sqrt{\sin\omega_1\sin\theta}\,\varphi^m_1(\theta,\nu_k)\\ \varphi^m_1(\theta,\nu_k)&=P^m_{\nu_k}(\cos\theta)l_1^*Q^m_{\nu_k}-Q^m_{\nu_k}(\cos\theta)l_1^*P^m_{\nu_k}\end{aligned} \tag{2.15}$$

Additionally, when account is taken of equality (2.14), we have

$$\varphi(\theta,\lambda_k)=\sqrt{\sin\omega_0\sin\theta}\,\frac{l_0^*Q^m_{\nu_k}\varphi^m_1(\theta,\nu_k)}{l_1^*Q^m_{\nu_k}\Gamma_m(\nu_k)}$$

$$\frac{\chi(\theta,\lambda_k)}{\varphi(\theta,\lambda_k)}=\sqrt{\frac{\sin\omega_1}{\sin\omega_0}}\frac{l_1^*Q^m_{\nu_k}}{l_0^*Q^m_{\nu_k}}$$

On taking account of the fact that, according to relation (2.12) and the equality $\nu = -1/2+i\sqrt{\lambda}$, the equality

$$\left.\frac{dW}{d\lambda}\right|_{\lambda=\lambda_k} = \left.\frac{dW}{d\nu}\frac{d\nu}{d\lambda}\right|_{\nu=\nu_k} = -\frac{\sqrt{\sin\omega_0\sin\omega_1}\,\tilde{\Delta}_k^m(\omega_0,\omega_1)}{(2\nu_k+1)\Gamma_m(\nu_k)} \tag{2.16}$$

$$\tilde{\Delta}_k^m(\omega_0,\ \omega_1) = \left.\frac{d\Delta_\nu^m}{d\nu}\right|_{\nu=\nu_k}$$

holds in accordance with formula (1.6.5) from [3], we arrive at the expansion

$$f(\theta) = -\sum_{k=0}^{\infty}\frac{\sqrt{\sin\theta}\varphi_1^m(\theta,\nu_k)}{\sigma_{mk}(\omega_0,\ \omega_1)}\int_{\omega_0}^{\omega_1}\sqrt{\sin\psi}\varphi_1^m(\psi,\nu_k)f(\psi)d\psi \tag{2.17}$$

where

$$\frac{1}{\sigma_{mk}(\omega_0,\omega_1)} = (2\nu_k+1)\frac{l_0^*Q_{\nu_k}^m}{l_1^*Q_{\nu_k}^m\cdot\Gamma_m(\nu_k)\tilde{\Delta}_k^m(\omega_0,\omega_1)} \tag{2.18}$$

As previously [3], it can be shown that expansion (2.17) holds for any function from $L([\omega_0,\ \omega_1])$ and behaves as regards its convergence in the same way as a conventional Fourier series. In particular, converges to $[f(\theta+0)+f(\theta+0)]/2$ if the function $f(\theta)$ has a bounded variation in the neighbourhood of the point θ.

It is obvious by construction that the eigenfunction $\varphi_1^m(\theta,\ \nu_k)$ satisfies $(-1/4-\lambda = \nu^2+\nu)$ the Legendre equation [4].

$$-\nabla_m\varphi_1^m(\theta,\nu_k)+\nu_k(\nu_k+1)\varphi_1^m(\theta,\nu_k) = 0,\quad \omega_0<\theta<\omega_1 \tag{2.19}$$

and the boundary condition

$$l_j^*\varphi_1^m(\theta,\nu_k) = 0,\quad j = 0,1 \tag{2.20}$$

The result obtained can be interpreted as follows. We introduce the function $g(\theta) = (\sin\theta)^{-1/2}f(\theta)$ for which the integral

$$\int_{\omega_0}^{\omega_1}\sqrt{\sin\theta}\,|g(\theta)|\,d\theta \tag{2.21}$$

exists.

Expansion (2.17) can then be treated as the inversion formula

$$g(\theta) = -\sum_{k=0}^{\infty}\frac{g_k^m\varphi_1^m(\theta,\nu_k)}{\sigma_{mk}(\omega_0,\omega_1)} \tag{2.22}$$

for the integral transform

$$g_k^m = \int_{\omega_0}^{\omega_1}\sin\theta\varphi_1^m(\theta,\nu_k)g(\theta)d\theta \tag{2.23}$$

We will now establish the relation between the eigenfunction (2.15), constructed for boundary-value problem (2.4) or boundary-value problem (2.19), (2.20) and the eigenfunctions of boundary-value problems (2.1) in cases a, b and c (we shall call these problems $(2.21)_a$, $(2.21)_b$ and $(2.21)_c$, respectively). We obtain the eigenfunction $\varphi_a^m(\theta,\ \nu_k)$ of boundary-value problem $(2.1)_a$ from (2.15), if, in (2.11) and boundary condition (2.20), we take $\alpha_j = 0$ $(j = 0,\ 1)$. As a result, we shall have the function

$$\varphi_a^m(\theta,\nu_k) = P_{\nu_k}^m(\cos\theta)Q_{\nu_k}^m(\cos\omega_1)-P_{\nu_k}^m(\cos\omega_1)Q_{\nu_k}^m(\cos\theta) \tag{2.24}$$

which satisfies the differential equation from (2.19) and the boundary conditions

$$\varphi_a^m(\omega_j, \nu_k) = 0, \quad j = 0, 1 \tag{2.25}$$

while the numbers ν_k have to be found from Eq. (2.3) which, in the case in question if one introduces the notation

$$\Omega_\nu^{l,n}(\omega_0, \omega_1) \equiv \Omega_\nu^{l,n} = P_\nu^l(\cos\omega_1)Q_\nu^n(\cos\omega_0) - P_\nu^n(\cos\omega_0)Q_\nu^l(\cos\omega_1) \tag{2.26}$$

can be written in the form

$$\Omega_{\nu_k}^{m,m}(\omega_0, \omega_1) = 0, \quad k = 0, 1, 2, \ldots \tag{2.27}$$

Here, formula (2.18) is also changed and acquires the form

$$\frac{1}{\sigma_{mk}^a(\omega_0, \ \omega_1)} = (2\nu_k + 1)\frac{Q_{\nu_k}^m(\cos\omega_0)}{Q_{\nu_k}^m(\cos\omega_1)}\left[\left.\frac{\partial\Omega_\nu^{m,m}(\omega_0, \omega_1)}{\partial\nu}\right|_{\nu=\nu_k}\right]^{-1} \tag{2.28}$$

In order to obtain the eigenfunction $\varphi_b^m(\theta, \nu_k)$ of boundary-value problem $(2.1)_b$, we divide the boundary conditions in (2.2) by $\sin\alpha$. We see from this that they convert into the boundary conditions of problems $(2.1)_b$ if it is taken that

$$\operatorname{ctg}\alpha_j + \tfrac{1}{2}\operatorname{ctg}\omega_j = h_j, \quad \operatorname{ctg}\alpha_j = h_j - \tfrac{1}{2}\operatorname{ctg}\omega_j, \quad j = 0, 1 \tag{2.29}$$

In this case, the functionals l_j^* become the functionals l_j^h, that is,

$$l_j^* P_\nu^m \equiv l_j^h P_\nu^m = \frac{dP_\nu^m(\cos\omega_j)}{d\omega_j} + h_j P_\nu^m(\cos\omega_j) \tag{2.30}$$

and, on the basis of relation (2.15), we obtain the function

$$\varphi_b^m(\theta, \nu_k) = P_{\nu_k}^m(\cos\theta) l_1^h Q_{\nu_k}^m - Q_{\nu_k}^m(\cos\theta) l_1^h Q_{\nu_k}^m \tag{2.31}$$

This function will satisfy the boundary conditions

$$\varphi_b^\bullet(\omega_j, \nu_k) + h_j\varphi_b(\omega_j, \nu_k) = 0, \quad j = 0, 1 \tag{2.32}$$

The numbers ν_k have to be found form Eq. (2.13). If account is taken of the notation (2.26) and use is made of the equality

$$dP_\nu^m(\cos\theta)/d\theta = P_\nu^{m+1}(\cos\theta) + m\operatorname{ctg}\theta P_\nu^m(\cos\theta) \tag{2.33}$$

(which also holds for $Q_\nu^m(\cos\theta)$) which follows from formula 3.6.1 (6) in [4], Eq. (2.13) takes the form

$$\Omega_{\nu_k,h}^m(\omega_0, \omega_1) = 0, \quad k = 0, 1, 2, \ldots \tag{2.34}$$

where

$$\Omega_{\nu,h}^m(\omega_0, \omega_1) = \Omega_\nu^{m+1,m+1} + (m\operatorname{ctg}\omega_0 + h_0)\Omega_\nu^{m+s,m} + (m\operatorname{ctg}\omega_1 + h_1)\Omega_\nu^{m,m+1} +$$
$$+[m^2\operatorname{ctg}\omega_0\operatorname{ctg}\omega_1 + m(h_0\operatorname{ctg}\omega_1 + h_1\operatorname{ctg}\omega_0) + h_0 h_1]\Omega_\nu^{m,m} \tag{2.35}$$

In the given case, formula (2.18) takes the form

$$\frac{1}{\sigma_{mk}^h(\omega_0, \omega_1)} = (2\nu_k + 1)\frac{l_0^h Q_{\nu_k}^m}{l_1^h Q_{\nu_k}^m \Gamma_m(\nu_k)}\left[\left.\frac{\partial\Omega_{\nu,h}^m}{\partial\nu}\right|_{\nu=\nu_k}\right]^{-1} \tag{2.36}$$

We obtain the eigenfunction $\varphi_c^m(\theta, \nu_k)$ of boundary-value problem $(2.1)_c$ from (2.31) on putting $h_j = 0$ ($j = 0, 1$) there, which leads to the formula

$$\varphi_c^m(\theta, \nu_k) = P_{\nu_k}^m(\cos\theta)\frac{dQ_{\nu_k}^m(\cos\omega_1)}{d\omega_1} - Q_{\nu_k}^m(\cos\theta)\frac{dP_{\nu_k}^m(\cos\omega_0)}{d\omega_0} \tag{2.37}$$

This function satisfies the boundary condition

$$\varphi_c^{\bullet}(\omega_j, \nu_k) = 0, \quad j = 0, 1 \tag{2.38}$$

and the numbers ν_k have to be found from the equation

$$\Delta_{\nu_k}^m \equiv \Omega_{\nu_k,0}^m(\omega_0, \omega_1) = 0, \quad k = 0, 1, 2, \ldots \tag{2.39}$$

We obtain an expression for $\Omega_{\nu_k,0}^m(\omega_0, \omega_1)$ from (2.35) by putting $h_0 = h_1 = 0$ there. In this case, formula (2.18) is written as

$$\frac{1}{\sigma_{m,k}^c(\omega_0, \omega_1)} = (2\nu_k + 1)\frac{dQ_{\nu_k}^m(\cos\omega_0)}{d\omega_0}\left[\frac{dQ_{\nu_k}^m(\cos\omega_1)}{d\omega_1}\frac{\partial}{\partial\nu}\Omega_{\nu,0}^m\Big|_{\nu=\nu_k}\right]^{-1} \tag{2.40}$$

Hence, integral transformation, which are based on Sturm–Liouville problems $(2.21)_a$, $(2.21)_b$ and $(2.21)_c$ and which hold for functions having a finite integral (2.21), can be written, according to relations (2.22) and (2.23), in the form

$$g_k^m = \int_{\omega_0}^{\omega_1} \sin\theta\varphi_e^m(\theta, \nu_k)g(\theta)d\theta, \quad e = a, b, c \tag{2.41}$$
$$g(\theta) = -\sum_{k=0}^{\infty}\frac{g_k^m\varphi_e^m(\theta, \nu_k)}{\sigma_{mk}^e(\omega_0, \omega_1)}, \quad \omega_0 \leqslant \theta \leqslant \omega_1$$

3. SPECIAL CASES OF THE INTEGRAL TRANSFORMATION OBTAINED

If we confine ourselves solely to the solution of the problem formulated above the application of the integral transformations (2.41) when $m = 0$ is found to be sufficient. However, on changing to a non-hollow cone ($\omega_0 = 0$), transformations (2.41) cannot be applied directly since, in these transformations, it is necessary to take the limit as ($\omega_0 \to 0$). First, we satisfy this passage to the limit expansion (2.17).

We start with the transcendental equation (2.13). By taking account of relations (2.12), (2.11) and formula 3.6.1 (2) from [4], it can be shown that the second term when $\omega_0 = 0$ in (2.13) is equal to zero and Eq. (2.13) becomes

$$l_1^* P_{\nu_k}^m = 0, \quad k = 0, 1, 2, \ldots$$
$$l_1^* P_\nu^m = \sin\alpha_1\frac{dP_\nu^m(\cos\omega)}{d\omega} + \left[\cos\alpha_1 + \frac{\operatorname{ctg}\omega}{2}\sin\alpha_1\right]P_\nu^m(\cos\omega) \tag{3.1}$$

or, by virtue of (2.33),

$$P_{\nu_k}^{m+1}(\cos\omega)\sin\alpha_1 + P_{\nu_k}^m(\cos\omega)[\cos\alpha_1 + (\operatorname{ctg}\omega\sin\alpha_1)(m + 1/2)] = 0, \quad k = 0, 1, 2, \ldots \tag{3.2}$$

According to relations (3.1) and (2.15), we arrive at the formula

$$\varphi_1^m(\theta, \nu_k) = l_1^* Q_{\nu_k}^m P_{\nu_k}^m(\cos\theta) \tag{3.3}$$

We now determine what the expression $\Delta_k^m(\omega_0, \omega_1)$ becomes when $\omega_0 \to 0$. According to relations (2.16) and (2.12), the equality

$$\frac{\partial \Delta_\nu^m}{\partial \nu} = l_0^* Q_\nu^m l_1^* \frac{\partial P_\nu^m}{\partial \nu} + l_1^* P_\nu^m l_0^* \frac{\partial Q_\nu^m}{\partial \nu} - l_0^* P_\nu^m l_1^* \frac{\partial Q_\nu^m}{\partial \nu} - l_1^* Q_\nu^m l_0^* \frac{\partial P_\nu^m}{\partial \nu} \tag{3.4}$$

has to be considered when $\nu = \nu_k$. By virtue of the behaviour of Legendre functions [4] in the neighbourhood of unity, the principal contribution to $\Delta_k^m(\omega_0, \omega_1)$ when $\omega_0 \to 0$ will be made by the first term from (3.4) and it is therefore necessary to take

$$\tilde{\Delta}_k^m(\omega_0, \omega_1) = l_0^* Q_\nu^m(\omega) \Delta_k^m(\omega), \quad \Delta_k^m(\omega) = \left.\frac{\partial l_1^* P_\nu^m}{\partial \nu}\right|_{\nu=\nu_k} \tag{3.5}$$

If expressions (3.3) and (3.5) are substituted into (2.17) and (2.18), we obtain

$$f(\theta) = -\sum_{k=0}^{\infty} \frac{\sqrt{\sin\theta}\, P_{\nu_k}^m(\cos\theta)}{\sigma_{mk}(\omega)} \int_0^{\omega} \sqrt{\sin\psi}\, P_{\nu_k}^m(\cos\psi) f(\psi) d\psi \tag{3.6}$$

where

$$\frac{1}{\sigma_{mk}(\omega)} = (2\nu_k + 1) \frac{l_1^* Q_{\nu_k}^m}{\Delta_k^m(\omega) \Gamma_m(\nu_k)} \tag{3.7}$$

If, as earlier, the function $g(\theta)$ is introduced, then expansion (3.6) can be looked upon as the inversion formula

$$g(\theta) = -\sum_{k=0}^{\infty} \frac{g_k^m(\omega)}{\sigma_{mk}(\omega)} P_{\nu_k}^m(\cos\theta) \tag{3.8}$$

for the integral transformation

$$g_k^m(\omega) = \int_0^{\omega} \sin\theta\, g(\theta) P_{\nu_k}^m(\cos\theta) d\theta \tag{3.9}$$

We see that the eigenfunction $\varphi_k^m(\theta) = P_{\nu_k}^m(\cos\theta)$ satisfies the Legendre differential equation (2.19) and boundary condition (3.1). This easily enables us to establish the relation between this function and the eigenfunctions of boundary-value problems (2.1), if they are assumed to be defined in the interval $(0, \omega)$, that is, when the boundary $\theta = \omega_0$ disappears and $\omega_1 = \omega$ and one can put $h_0 = 0$, $h_1 = h$, $\omega_s = \omega$, $\alpha_1 = \alpha$. Then, according to relations (3.1), the function $\varphi_k^m(\theta)$ will be an eigenfunction of problem $(2.1)_a$ when $\omega_0 = 0$, $\omega_1 = \omega$ if we put $\alpha_1 = \alpha = 0$ in (3.1) and the numbers ν_k are found from the transcendental equation

$$P_{\nu_k}^m(\cos\omega) = 0, \quad k = 0, 1, 2, \ldots \tag{3.10}$$

The expansion formula (3.8) and (3.9) have been given previously in a some what different form [5, 6] for this special case.

The function $P_{\nu_k}^m(\cos\theta)$ will also be an eigenfunction of problem $(2.1)_b$ when $\omega_0 = 0$, $\omega_1 = \omega$ if the numbers ν_k are found from the transcendental equation

$$P_{\nu_k}^{m+1}(\cos\omega) + (h + m\,\mathrm{ctg}\,\omega) P_{\nu_k}^m(\cos\omega) = 0, \quad k = 0, 1, 2, \ldots \tag{3.11}$$

as can be shown by putting $\mathrm{ctg}\,\alpha_1 = h - 1/2\,\mathrm{ctg}\,\omega$ in (3.1) and using (2.33).

The function $\varphi_k^m(\theta)$ will also be an eigenfunction of problem $(2.1)_c$ when $\omega_0 = 0$, $\omega_1 = \omega$, if the numbers ν_k are found from the transcendental equation

$$P_{\nu_k}^{m+1}(\cos\omega) + m\,\mathrm{ctg}\,\omega P_{\nu_k}^m(\cos\omega) = 0, \quad k = 0, 1, 2, \ldots \tag{3.12}$$

which follows from (3.11) if one puts $h = 0$ there.

It is important to note that the transcendental equations (3.10) and (3.12) admit of explicit solutions when $\omega = \pi/2$.

We shall now prove this. For example, putting $\omega = \pi/2$ in (3.10) and using formulae 3.4 (20) and 1.2 (7) from [4], we reduce Eq. (3.10) to the form

$$2^m\sqrt{\pi}[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m-\tfrac{1}{2}\nu_k)\Gamma(1-\tfrac{1}{2}m+\tfrac{1}{2}\nu_k)]^{-1}=0 \tag{3.13}$$

from which we recognize two series of values $\nu_k^0 = 2k - m + 1$, $\nu_k^1 = -2k + m - 2$ which make the left-hand side of equality (3.13) vanish. Here, the eigenfunction

$$\varphi_k^m(\theta) = P_{\nu_k^0}^m(\cos\theta) = P_{2k-m+1}^m(\cos\theta) \tag{3.14}$$

corresponds to the first series of values, ν_k^0 but, according to formula 3.6.1 (2) from [4], we have for integral indices

$$P_l^m(z) = 0, \quad m > l \tag{3.15}$$

and, hence, eigenfunction (3.16) will not be zero solely when $k \geqslant m$. For this reason, the eigenfunctions corresponding to the values ν_k^1 are equal to zero for any value of $k \geqslant 0$, and the values of ν_k therefore have to be discarded. In order to write out expansion (3.6) or (3.8), (3.9) for this particular system, it is necessary to calculate $\sigma_{mk}(\pi/2)$ using formula (3.7) but in the case under consideration, we have

$$\begin{aligned} &\nu_k = 2k-m+1, \quad l_1^* Q_{\nu_k}^m = Q_{2k-m+1}^m(0) = \Delta_k^m\left(\frac{\pi}{2}\right) = \\ &= \left.\frac{\partial P_\nu^m(0)}{\partial\nu}\right|_{\nu=2k-m+1} = \frac{(-1)^{k+1}2^{m-1}\sqrt{\pi}k!}{\Gamma(k-m+\tfrac{3}{2})} \\ &\Gamma_m(2k-m+1) = \frac{2^{2m}k!\Gamma(k+\tfrac{3}{2})}{\Gamma(k-m+\tfrac{3}{2})\Gamma(k-m+1)} \end{aligned} \tag{3.16}$$

In order to obtain formula (3.16), equalities 3.4 (21) and and 3.4 (20) from [4] have been taken into account.

On the basis of relations (3.7), (3.4) and (3.6), expansion (3.8) becomes

$$g(\theta) = -\sum_{k=m}^{\infty}\frac{g_k^m(\pi/2)(4k-2m+3)(k-m)!\Gamma(k-m+\tfrac{3}{2})P_{2k-m+1}^m(\cos\theta)}{2^{2m}k!\Gamma(k+\tfrac{3}{2})} \tag{3.17}$$

In the special case when $m = 0$, we have

$$g(\theta) = -\sum_{k=0}^{\infty} g_k\left(\frac{\pi}{2}\right)(4k+3)P_{2k+1}(\cos\theta), \quad g_k\left(\frac{\pi}{2}\right) = \int_0^{\pi/2}\sin\theta g(\theta)P_{2k+1}(\cos\theta)d\theta \tag{3.18}$$

which is identical to the well-known result in [5].

We will now consider Eq. (3.12) when $\omega = \pi/2$. Like Eq. (3.10), Eq. (3.12) becomes Eq. (3.13), in which m has to be replaced by $m + 1$, which leads to the expansion ($\nu_k = 2k - m$)

$$\begin{aligned} &g(\theta) = -\sum_{k=m}^{\infty}\frac{g_k^m(\pi/2)(4k-2m+1)}{\Gamma_m(2k-m)}P_{2k-m}^m(\cos\theta) \\ &\Gamma_m(2k-m) = \frac{2^{2m}k!\Gamma(k+\tfrac{1}{2})}{\Gamma(k-m+\tfrac{1}{2})\Gamma(k-m+1)} \end{aligned} \tag{3.19}$$

which, in the special case when $m = 0$, gives the result known from [5].

4. THE SOLUTION OF THE PROBLEM

We will now apply integral transformation (2.41) to boundary-value problem (1.2), (1.3). In order to satisfy conditions (1.3), it is sufficient, by formulae (1.8) and (1.9), to require that

$$\Psi'(r,\omega_j) = \Omega'(r,\omega_j) = 0, \quad j = 0,1 \tag{4.1}$$

These conditions dictate that integral transformation (2.41) when $m = 0$ and $e = c$ be applied to the equations

$$\Delta\Psi = 0, \quad \Delta\Omega = 0, \quad a < r < b, \quad \omega_0 < \theta < \omega_1 \tag{4.2}$$

The eigenfunction (2.37), which satisfies the Legendre equation (2.19), which we write in the form,

$$\varphi_c(\theta, \nu_k) = P_{\nu_k}(\cos\theta)\frac{dQ_{\nu_k}(\cos\omega_1)}{d\omega_1} - Q_{\nu_k}(\cos\theta)\frac{dP_{\nu_k}(\cos\omega_0)}{d\omega_0} \tag{4.3}$$

is the kernel of the above-mentioned transformation.

The numbers ν_k have to be found from Eq. (2.39) which can now be written in the form

$$\begin{aligned} &\Omega_\nu^1\Big|_{\nu=\nu_k} = 0, \quad k = 0, 1, 2, \ldots \\ &\Omega_\nu^1 = P_\nu^1(\cos\omega_1)Q_\nu^1(\cos\omega_0) - P_\nu^1(\cos\omega_0)Q_\nu^1(\cos\omega_1) \end{aligned} \tag{4.4}$$

According to relation (2.38), the function (4.3) satisfies the condition

$$\varphi_c(\omega_j, \nu_k) = 0, \quad j = 0, 1 \tag{4.5}$$

On applying the above-mentioned integral transformation to Eqs (4.2), taking account of conditions (4.1) and (4.5) and solving the resulting differential equations for the transform of the functions Ψ and Ω, we find

$$\begin{Vmatrix} \Psi_k(r) \\ \Omega_k(r) \end{Vmatrix} = \int_{\omega_0}^{\omega_1} \sin\theta \begin{Vmatrix} \Psi(r,\theta) \\ \Omega(r,\theta) \end{Vmatrix} \varphi_c(\theta, \nu_k)d\theta = \begin{Vmatrix} C_k^0 r^{\nu_k} + D_k^0 r^{-(\nu_k+1)} \\ C_k^1 r^{\nu_k} + D_k^1 r^{-(\nu_k+1)} \end{Vmatrix} \tag{4.6}$$

where C_k^j, D_k^j ($j = 0, 1$) are arbitrary constants which have to be found by satisfying boundary conditions (1.2).

First, we satisfy the conditions

$$\tau_r(b, \theta) = \tau_r(a, \theta) = 0, \quad \omega_0 \leqslant \theta \leqslant \omega_1 \tag{4.7}$$

In order to do this, it is sufficient to require that

$$A(b, \theta) = A(a, \theta) = 0, \quad \omega_0 \leqslant \theta \leqslant \omega_1$$

and to apply integral transformation (4.6) which leads to the equalities

$$A_k(b) = A_k(a) = 0, \quad k = 0, 1, 2, \ldots \tag{4.8}$$

where

$$\begin{aligned} &A_k(r) = \int_{\omega_0}^{\omega_1} \sin\theta A(r,\theta)\varphi_c(\theta, \nu_k)d\theta = \\ &= (3 - 4\mu)[r^{-1}\Omega_k(r)]' + \Omega_k''(r) + \mu r^2\Psi_k'' + (1 + 2\mu)r\Psi_k'(r) - \\ &-(1 - 2\mu)\Psi_k(r) + (1 - \mu)(\nu_k^2 + \nu_k)\Psi_k(r) \end{aligned} \tag{4.9}$$

Substituting expressions (4.6) here and satisfying conditions (4.8), we obtain the equations

$$\begin{aligned} &C_k^0 e_1(2\nu_k + 3)p_2^0(\nu_k) + C_k^1 e_1(2\nu_k + 1)p_2^1(\nu_k) + D_k^0 e_2(2)q_2^0(\nu_k) = 0 \\ &C_k^0 e_1(2)e_2(2\nu_k + 1)e_1(2)p_2^0(\nu_k) - D_k^0 e_1(2\nu_k - 1)e_2(2)q_2^0(\nu_k) - \\ &-D_k^1 e_1(2\nu_k + 1)q_2^1(\nu_k) = 0 \end{aligned} \tag{4.10}$$

Here,

$$
\begin{aligned}
&e_1(x) = b^x - a^x, \quad e_2(x) = (ab)^x \\
&p_2^0(x) = x^2 + 2x - (1-2\mu), \quad p_2^1(x) = x^2 + 2(1-2\mu)x - (3-4\mu) \\
&q_2^0(x) = x^2 - 2(1-\mu), \quad q_2^1(x) = x^2 - 4\mu x - 4(1-2\mu)
\end{aligned}
\tag{4.11}
$$

We now satisfy the remaining boundary conditions from (1.2), that is,

$$
(1-2\mu)\sigma_2(b,\theta) = -(1-2\mu)p(\theta), \quad \sigma_r(a,\theta) = 0, \quad \omega_0 \leqslant \theta \leqslant \omega_1
$$

We again apply integral transformation (4.6) to the conditions stated. As a result, after taking account of formulae (1.10) and (4.6), we arrive at equations which can be obtained from Eqs (4, 10) if the following substitutions are made in them: $A\mu(\nu_k)$ and $B\mu(\nu_k)$ are substituted instead of $p_2^0(\nu_k)$ and $p_2^1(\nu_k)$ and $q_2^0(\nu_k)$ and $q_2^1(\nu_k)$ are substituted instead of $C_\mu(\nu_k)$ and $D_\mu(\nu_k)$ respectively and, on the right-hand side of the first equation, the expression $(1 - 2\mu)p_k b^{\nu_k+3}$ is substituted instead of zero and, on the right-hand side of the second equation, $(1 - 2\mu)p_k b^{\nu_k+3}$ is substituted instead of zero. Here, the symbols in the above substitutions are interpreted as

$$
\begin{Vmatrix} A_\mu(\nu_k) \\ B_\mu(\nu_k) \end{Vmatrix} = \sum_{j=0}^{3} \begin{Vmatrix} a_j(\mu) \\ b_j(\mu) \end{Vmatrix} \nu_k^j, \quad \begin{Vmatrix} C_\mu(\nu_k) \\ D_\mu(\nu_k) \end{Vmatrix} = \sum_{j=0}^{3} \begin{Vmatrix} c_j(\mu) \\ d_j(\mu) \end{Vmatrix} \nu_k^j
\tag{4.12}
$$

$$
\begin{aligned}
&p_k = \int_{\omega_0}^{\omega_1} \sin\theta p(\theta)\varphi_c(\theta,\nu_k)d\theta \\
&a_3(\mu) = b_3(\mu) = -c_3(\mu) = -d_3(\mu) = -c_2(\mu)/2 = 1-2\mu \\
&a_2(\mu) = b_0(\mu) = c_0(\mu) = 0, \quad b_2(\mu) = \tfrac{1}{2}d_0(\mu) = 2(4\mu^2 - 4\mu + 1) \\
&a_0(\mu) = 2(\mu+1), \quad a_1(\mu) = 4\mu(\mu+1) - 3 \\
&b_1(\mu) = 10\mu - 3, \quad c_1(\mu) = 2\mu(1-2\mu) \\
&d_1(\mu) = 4(6\mu^2 - 5\mu + 1), \quad d_2(\mu) = 8\mu^2 - 2\mu + 1
\end{aligned}
$$

Solving the system of algebraic equations consisting of Eqs (4.10) and the two equations described above, we obtain

$$
\begin{aligned}
&\begin{Vmatrix} -C_k^0 \\ D_k^0 \end{Vmatrix} = \frac{(1-2\mu)p_k}{\Lambda_c^\mu(\nu_k)} \begin{Vmatrix} C_c^\mu(\nu_k) \\ D_c^\mu(\nu_k) \end{Vmatrix} \\
&\begin{Vmatrix} C_k^1 p_2^1(\nu_k) \\ D_k^1 q_2^1(\nu_k) \end{Vmatrix} = \frac{(1-2\mu)p_k}{\Lambda_c^\mu(\nu_k)e_1(2\nu_k+1)} \times \\
&\times \left\{ C_c^\mu(\nu_k)p_2^0(\nu_k) \begin{Vmatrix} e_1(2\nu_k+3) \\ e_1(2)e_2(2\nu_k+1) \end{Vmatrix} + D_c^\mu(\nu_k)q_2^0(\nu_k) \begin{Vmatrix} -e_1(2) \\ e_1(2\nu_k-1)e_2(2) \end{Vmatrix} \right\}
\end{aligned}
\tag{4.13}
$$

Here, it is assumed that

$$
\begin{aligned}
&\Lambda_c^\mu(\nu_k) = A_{11}^\mu(\nu_k)A_{22}^\mu(\nu_k)e_1(2\nu_k+3)e_1(2\nu_k-1)e_2(2) + A_{12}^\mu(\nu_k)A_{21}^\mu(\nu_k)e_1^2(2)e_2(2\nu_k+1) \\
&\begin{Vmatrix} C_c^\mu(\nu_k) \\ D_c^\mu(\nu_k) \end{Vmatrix} = b^{\nu_k+3}p_2^1(\nu_k) \begin{Vmatrix} A_{22}^\mu(\nu_k)e_1(2\nu_k-1)e_2(2) \\ -A_{21}^\mu(\nu_k)e_1(2)e_2(2\nu_k+3) \end{Vmatrix} + \\
&+ a^{\nu_k-2}e_2(\nu_k+3)q_2^1(\nu_k) \begin{Vmatrix} A_{12}^\mu(\nu_k)e_1(2) \\ A_{11}^\mu(\nu_k)e_1(2\nu_k+3) \end{Vmatrix} \\
&\begin{Vmatrix} A_{11}^\mu(\nu_k) \\ A_{12}^\mu(\nu_k) \end{Vmatrix} = p_2^1(\nu_k) \begin{Vmatrix} A_\mu(\nu_k) \\ C_\mu(\nu_k) \end{Vmatrix} - B_\mu(\nu_k) \begin{Vmatrix} p_2^0(\nu_k) \\ q_2^0(\nu_k) \end{Vmatrix}
\end{aligned}
\tag{4.14}
$$

$$\left\| \begin{matrix} A_{21}^{\mu}(\nu_k) \\ A_{22}^{\mu}(\nu_k) \end{matrix} \right\| = q_2^1(\nu_k) \left\| \begin{matrix} A_{\mu}(\nu_k) \\ C_{\mu}(\nu_k) \end{matrix} \right\| - D_{\mu}(\nu_k) \left\| \begin{matrix} p_2^0(\nu_k) \\ q_2^0(\nu_k) \end{matrix} \right\|$$

Hence, all of the unknown coefficients have been found. Substituting their values into (4.6) and inverting the transforms using inversion formula (2.41), putting $m = 0$ and $e = c$ in this formula, we obtain

$$\left\| \begin{matrix} \Psi(r,\theta) \\ \Omega(r,\theta) \end{matrix} \right\| = -\sum_{k=0}^{\infty} \frac{\varphi_c(\theta,\nu_k)}{\sigma_{0k}^c(\omega_0,\omega_1)} \left\| \begin{matrix} C_k^0 r^{\nu_k} + D_k^0 r^{-(\nu_k+1)} \\ C_k^1 r^{\nu_k} + D_k^1 r^{-(\nu_k+1)} \end{matrix} \right\|, \quad \omega_0 \leqslant \theta \leqslant \omega_1 \tag{4.15}$$

where, by relations (2.40), (2.35) and (2.26),

$$\frac{1}{\sigma_{0k}^c(\omega_0,\omega_1)} = (2\nu_k + 1)\frac{dQ_{\nu_k}(\cos\omega_0)}{d\omega_0}\left[\frac{dQ_{\nu_k}(\cos\omega_1)}{d\omega_1}\frac{d\Omega_\nu^1}{d\nu}\bigg|_{\nu=\nu_k}\right]^{-1} \tag{4.16}$$

Knowing the functions $\Psi(r, \theta)$ and $\Omega(r, \theta)$, we can then find the strain and stress fields, that is, we can completely solve the problem in question.

Remark. The result which has been obtained can be extended in an obvious manner to the case when a normal load is also applied over the second end of the cone being considered (that is, over the surface $r = a$). It is more difficult to extend it if a shear load $\tau_i(\theta)$ ($i = 0, 1$) is applied to both ends, $r = a = a_0$ and $r = b = a_1$, that is, instead of boundary conditions (4.7), we must satisfy the conditions

$$\tau_r(a_i,\theta) = A'(a_i,\theta) = \tau_i(\theta), \quad i = 0,1; \ \omega_0 \leqslant \theta \leqslant \omega_1 \tag{4.17}$$

One of the ways in which the results obtained above can be extended is as follows. Integration with respect to θ is carried out in relations (4.17) which leads to the equalities

$$A(a_i,\theta) = T_i(\theta) + B_i, \quad i = 0,1; \quad T_i(\theta) = \int_{\omega_0}^{\theta} \tau_i(\Psi)d\Psi \tag{4.18}$$

where

$$B_i = A(a_i, \omega), \quad i = 0, 1 \tag{4.19}$$

We apply integral transformation (4.6) to equalities (4.18), and as a result, instead of the homogeneous conditions (4.8), we obtain the following

$$A_k(a_i) = T_{ik} + B_i \cdot \gamma_k, \quad \gamma_k = \int_{\omega_0}^{\omega_1} \sin\theta\varphi_c(\theta,\nu_k)d\theta \tag{4.20}$$

By satisfying these conditions in the same ways as conditions (4.8), we obtain formulae for the required coefficients of problem (4.13) which contain B_i. We then find $A(a_i, \omega)$($i = 0, 1$) using formulae (4.15) and (1.9). The resulting expressions are substituted into (4.19), which leads to a system of two algebraic equations from which we find B_i ($i = 0, 1$).

The solutions of some interesting special cases of the problem are contained in formulae (4.13)–(4.16).

For example, in the case when there is no void in the cone, that is, $\omega_0 = 0$ and $\omega_0 = \omega$, formulae (4.13)–(4.16), which determine the solution, in this case also hold with the following correction, according to section 3

$$\varphi_c(\theta,\nu_k) = P_{\nu_k}^1(\cos\theta), \quad \frac{1}{\sigma_{0k}^c(\omega_0,\omega_1)} = \frac{(2\nu_k + 1)Q_{\nu_k}^1(\cos\omega)}{[\partial P_\nu^1(\cos\omega)/\partial\nu]_{\nu=\nu_k}}$$

and the numbers ν_k have to be found form the equation $P_{\nu_k}^1(\cos\omega) = 0$ from which, in particular, it follows, by virtue of (3.15), that $\nu_0 = 0$.

If, in the problem under consideration, apart from $\omega_0 = 0$ it is additionally assumed that $\omega_1 = \omega = \pi/2$, then the resulting formulae (4.13)–(4.15) give the exact solution of the problem of the stressed state of a thin-walled spherical cupola supported on an absolutely rigid, smooth base. In this case, it is necessary to use integral transformation (3.9) with $m = 0$ instead of integral transformation (4.6). Then, instead of (4.15), we will have

$$\left\|\begin{matrix}\Psi(r,\theta)\\ \Omega(r,\theta)\end{matrix}\right\| = -\sum_{k=0}^{\infty}(4k+1)P_{2k}(\cos\theta)\left\|\begin{matrix}C_k^0 r^{2k} + D_k^0 r^{-(2k+1)}\\ C_k^1 r^{2k} + D_k^1 r^{-(2k+1)}\end{matrix}\right\| \tag{4.21}$$

Formulae (4.13) and (4.16) for the coefficients C_k^j, D_k^j $(j = 0, 1)$ still hold as well as formulae (4.11) and (4.12) with $\nu_k = 2k$ $(k = 0, 1, 2, \ldots)$ substituted into them and also

$$p_k = \int_0^{\pi/2}\sin\theta p(\theta)P_{2k}(\cos\theta)d\theta \tag{4.22}$$

If $\omega_1 = \pi/2$ is retained but it is not assumed that $\omega_0 = 0$, then formulae (4.13)–(4.16) with the obvious correction give the solution of the problem of the stressed state of a thin-walled spherical cupola with a conical incision at the centre of the cupola.

Taking the limit as $b \to \infty$ in formulae (4.13)–(4.16), (4.11), we obtain the solution of the problem in the case of a semi-infinite hollow cone while, taking the limit as $a \to 0$, we obtain the solution in the case of a finite hollow cone with a point.

We now consider the case when $a \to 0$. By formulae (4.11), $e_1(x) = b^x$, $e_2(x) = 0$.

Instead of relations (4.13) and (4.15), we will have

$$\left\|\begin{matrix}-C_k^0\\ C_k^1\end{matrix}\right\| = \frac{(1-2\mu)p_k}{A_{11}^{\mu}(\nu_k)b^{\nu_k}}\left\|\begin{matrix}p_2^1(\nu_k)\\ p_2^0(\nu_k)b^2\end{matrix}\right\|, \quad D_k^0 = D_1^k = 0 \tag{4.23}$$

$$\left\|\begin{matrix}\Psi(r,\theta)\\ \Omega(r,\theta)\end{matrix}\right\| = -\sum_{k=0}^{\infty}\frac{\varphi_c(\theta,\nu_k)}{\sigma_{0k}^c(\omega_0,\omega_1)}\left\|\begin{matrix}C_k^0 r^{\nu_k}\\ C_k^1 r^{\nu_k}\end{matrix}\right\|, \quad \omega_0 \leqslant \theta \leqslant \omega_1 \tag{4.24}$$

Finally, putting $\omega_0 = 0$, $\omega_1 = \omega = \pi/2$, we obtain the exact solution of the problem of the stressed state of a hemisphere supported on an absolutely rigid, smooth base. However, it is necessary here to use formula (4.21), instead of formula (4.24), to put $\nu_k = 2k$ $(k = 0, 1, 2, \ldots)$ in (4.21) and to take relation (4.22) into account. The exact solution of a similar problem but for a Ponderable hemisphere has been obtained by B. F. Bondareva using another method [7].

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